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The fourth power of any number is divisible by 16, if the number is even, by 16 with remainder 1 if the number is odd.

We consider two cases: (I) $n$ not divisible by 3, (II) $n$ divisible by 3.

**Case I.** If $n$ is even, $n - 3$ is odd, and the two are prime to each other. Since their product is twice a fourth power, $n - 3$ is a fourth power. Consequently $n - 3$ is divisible by 16 with remainder 1, by $n$ with remainder 4, and by $n/2$ with remainder 2 or 10; in neither case can $n/2$ be a fourth power, so that $n(n - 3)/2$ cannot be a fourth power either.

If $n$ is odd, it must be a fourth power by the same reasoning as before. $n - 3$ has the remainder 14, and $(n - 3)/2$, has the remainder 7 or 15, when divided by 16; so that $(n - 3)/2$ cannot be a fourth power, nor can $n(n - 3)/2$.

**Case II.** If $n$ is divisible by 3, $n - 3$ is also, and so is the fourth power, so that we have the equation $n(n - 3) = 2\cdot 3^4\cdot k^4$. Either $n$ or $n - 3$ is divisible by $3^3$ and the other one by 3. Calling the one $3^3\beta$ the other will be $3^3\beta \pm 3$, the sign depending on which is divisible by $3^3$. Our equation takes the form $3^3\beta(3^3\beta \pm 1) = 2\cdot 3^4\cdot k^4$, or $\beta(3^2\beta \pm 1) = 2\cdot k^4$.

If $\beta$ is odd, it must be a fourth power, and $(3^2\beta \pm 1)$ twice a fourth power. But if $\beta \equiv 1 \pmod{16}$, $3^2\beta \pm 1 \equiv 8$ or $10 \pmod{16}$, so that it could not be twice a fourth power.

If $\beta$ is even, $\beta/2$ and $3^2\beta \pm 1$ must both be fourth powers. Here, again, two cases are possible: $\beta/2$ even and $\beta/2$ odd. In the latter case $\beta/2 \equiv 1 \pmod{16}$, or $\beta \equiv 2 \pmod{16}$, so that $3^2\beta \pm 1 \equiv 3$ or $1 \pmod{16}$, from which we deduce at once that only the $-$ sign is admissible. But $3^2\beta - 1 =$ a fourth power, and any fourth power if divided by 3 must yield the remainder $+1$ or 0. Consequently $3^2\beta - 1$ cannot be a fourth power, if $\beta/2$ is odd.

It remains to consider the case $\beta/2$ even. Just as before, we see that $\beta/2$ and $3^2\beta \pm 1$ must be fourth powers. Calling these $16b^4$ and $a^4$ respectively, (since $\beta/2$ is even, it must be the fourth power of an even integer) we derive the equations $288b^4 \pm 1 = a^4$. Since $a^4 \equiv 1 \pmod{16}$. the $+$ sign only is admissible. We now proceed to show the impossibility of this last equation,

Writing it in the form $288b^4 = a^4 - 1$, we see that for the right hand side to be divisible by 32, $a$ must be of the form $8\lambda \pm 1$. Substituting this value for $a$ and factoring the right-hand side, the equation becomes

$$288b^4 = (64\lambda^2 \pm 16\lambda + 2)(8\lambda \pm 2)8\lambda,$$

or, after division by 32,

$$9b^4 = (32\lambda^2 \pm 8\lambda + 1)(4\lambda \pm 1)\lambda.$$

It is easy to see that the three factors on the right are prime to each other: in fact

$$(32\lambda^2 \pm 8\lambda + 1) - (4\lambda \pm 1)^2 = 16\lambda^2,$$

so that any common factor of the first two must be a factor of $16\lambda^2$. Since the product of these three is a perfect square, each of them must be a square, and we have the three equations

$$\lambda = c^2,$$

$$4\lambda^2 \pm 1 = d^2,$$

$$32\lambda^2 \pm 8\lambda + 1 = e^2.$$

From the first two of these we deduce

$$4c^2 \pm 1 = d^2,$$

which is impossible, for the squares of two integers cannot differ by unity.

As we have examined all possible cases, we conclude that no polygon can have the number of its diagonals equal to a perfect fourth power.

**222.** (**October, 1914.**) **Proposed by A. H. HOLMES, Brunswick, Me.**

Find rational values for $m$ and $n$ such that $(m^2 + 1)/m^2 + (n^2 + 1)/n^2$ may be the square of an integer.

## SOLUTION BY ELIJAH SWIFT, University of Vermont.

Let $m = b/a$, $n = d/c$, where $a$, $b$, $c$, $d$, are all integers, and where $a$ is prime to $b$, and $c$ to $d$. Substituting these values, $\dfrac{b^2 + a^2}{b^2} + \dfrac{c^2 + d^2}{d^2}$ must be the square of an integer. If it is to be an integer at all, clearly $b$ must equal $d$, so that we have the equation

$$\frac{a^2 + c^2 + 2b^2}{b^2} = \text{a perfect square},$$

or

$$a^2 + c^2 = b^2 \, [\text{a perfect square} - 2] = b^2k,$$

where $k$ must have one of the values 2 (*i. e.*, $2^2 - 2$), 7, 14, 23, 34, 47, 62, 79, 98, $\cdots$.

If $k$ has one of the values 2, 34, 98, this equation possesses a solution, but not for any of the other values given. To show this and the method of obtaining the solution, it will be sufficient to treat the values $k = 2$, and $k = 7$. If $k = 7$, $a$ and $c$ must be prime to each other, and also to 7, as otherwise they would not be prime to $b$. Consequently each must be of the form $7\lambda \pm 1, 2$, or 3. But the sum of their squares cannot be divisible by 7 in that case, and the equation is impossible. If $k = 2$, let $x = a/b$ and $y = c/b$, so that the equation takes the form $x^2 + y^2 = 2$. An obvious solution is $x = 1$, $y = 1$. Any line through the point (1, 1) with rational slope will cut the circle $x^2 + y^2 = 2$ in a second rational point, and, conversely, taking the slope to be $\alpha/\beta$, where $\alpha$ and $\beta$ are integers, and solving for $x$ and $y$, we get the general solution of our equation to be

$$x = \frac{a}{b} = \frac{\alpha^2 - 2\alpha\beta - \beta^2}{\alpha^2 + \beta^2}, \qquad y = \frac{c}{b} = \frac{\beta^2 - 2\alpha\beta - \alpha^2}{\alpha^2 + \beta^2},$$

where $\alpha$ and $\beta$ are any integers. $m$ and $n$ are the reciprocals of $x$ and $y$ respectively.

A similar solution can readily be found in the other possible cases.

# QUESTIONS AND DISCUSSIONS.

[Send all Communications to U. G. MITCHELL, University of Kansas, Lawrence, Kans.]

## DISCUSSIONS.

### I. RELATING TO SOME DETERMINANTS CONNECTED WITH THE BERNOULLI NUMBERS.

BY K. P. WILLIAMS, Indiana University.

It is well known that the Bernoulli numbers $B_1$, $B_2$, $B_3$, $\cdots$ can be obtained from either of the following recurrence relations,

$${}_{2n+1}C_{2n}B_n - {}_{2n+1}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}_{2n+1}C_2B_1 = (-1)^{n-1}(n - \tfrac{1}{2}),$$

or

$${}_{2n+2}C_{2n}B_n - {}_{2n+2}C_{2n-2}B_{n-1} + \cdots (-1)^{n-1}_{2n+2}C_2B_1 = (-1)^{n-1}n,$$

where ${}_nC_r$ denotes the combinations of $n$ things $r$ at a time.[1] If we write out several of the equations we obtain the two series of relations

$$\begin{array}{ll}
{}_3C_2B_1 = \tfrac{1}{2}, & {}_4C_2B_1 = 1, \\
{}_5C_2B_1 - {}_5C_4B_2 = \tfrac{3}{2}, & {}_6C_2B_1 - {}_6C_4B_2 = 2, \\
{}_7C_2B_1 - {}_7C_4B_2 + {}_7C_6B_3 = \tfrac{5}{2}, & {}_8C_2B_1 - {}_8C_4B_2 + {}_8C_6B_3 = 3, \\
{}_9C_2B_1 - {}_9C_4B_2 + {}_9C_6B_3 - {}_9C_8B_4 = \tfrac{7}{2}, & {}_{10}C_2B_1 - {}_{10}C_4B_2 + {}_{10}C_6B_3 - {}_{10}C_8B_4 = 4; \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

Since either set will determine $B_1$, $B_2$, $B_3$, $\cdots$ it is obvious that the first $n$ equations in one set and the $n$th equation of the other set will form a system of $n + 1$ consistent equations in the $n$ quantities $B_1, B_2, \cdots, B_n$; thus the augmented determinant will be zero. For instance, if $n = 4$, we have

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[1] Chrystal, *Algebra*, Part II, p. 207 (first edition), or p. 231 (second edition, 1900).